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EXTENSION OF HURWITZ'S PROOF FOR THE TRANSCENDENCE OF e TO THE TRANSCENDENCE OF π.*

By Robert E. Moritz.

To show that π is transcendental it is sufficient to show that e cannot satisfy an equation of the form

$$C_0 + e^{\alpha_1} + e^{\alpha_2} + \cdots\cdots + e^{\alpha_n} = 0, \qquad [A]$$

where C_0 is an integer and the α's are associated algebraic numbers, that is, numbers which taken together constitute all the roots of some algebraic equation with integral coefficients.

For if π were algebraic, so would $i\pi$ be, that is, $i\pi$ would satisfy an equation with integral coefficients, as

$$a(x - \rho_1)\ (x - \rho_2) \cdots\cdots (x - \rho_m) = 0,$$

from which, since $e^{i\pi} = -1$, we should have

$$(1 + e^{\rho_1})\ (1 + e^{\rho_2}) \cdots\cdots (1 + e^{\rho_m}) = 1 + e^{\sigma_1} + e^{\sigma_2} + \cdots + e^{\sigma_r} = 0.$$

Here the σ's are sums of ρ's and are hence roots of some equation with integral coefficients. Of course the σ's may include zero values. Denote the σ's which do not vanish by $\alpha_1, \alpha_2, \ldots \alpha_n$. The last equation then assumes the form

$$C_0 + e^{\alpha_1} + e^{\alpha_2} + \cdots\cdots\cdots + e^{\alpha_n} = 0,$$

in which $C_0 - 1$ represents the number of the σ's which have vanished. C_0 is therefore an integer and the α's are algebraic numbers, satisfying an equation with integral coefficients, say

$$a_0 x^n + a_1 x^{n-1} + \cdots\cdots + a_{n-1} x + a_n = 0. \qquad (1)$$

Hurwitz,† in his proof for the transcendence of e, defines a function $F(x)$ by means of the equation

$$F(x) = f(x) + f'(x) + f''(x) + \cdots\cdots$$

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* This paper was read before the Chicago Section of the American Mathematical Society at the meeting of April 14, 1900.

† *Math. Annalen*, vol. 43 (1893), p. 220.

where $f(x)$ is an arbitrarily chosen integral rational function of x and the terms on the right hand side end with a constant term. The application of the law of the mean to the function $e^{-x}F(x)$ leads to the relation

$$F(x) - e^x F(0) = -xe^{(1-\theta)x} f(\theta x), \qquad 0 < \theta < 1.$$

Let us denote the right hand member of the last equation by ϵ_x and substitute successively for x the values $a_1, a_2, \ldots . a_n$. We get

$$\begin{aligned} F(a_1) - e^{a_1} F(0) &= \epsilon_{a_1}, \\ F(a_2) - e^{a_2} F(0) &= \epsilon_{a_2}, \\ \cdot \quad & \cdot \quad \cdot \\ \cdot \quad & \cdot \quad \cdot \\ \cdot \quad & \cdot \quad \cdot \\ F(a_n) - e^{a_n} F(0) &= \epsilon_{a_n}. \end{aligned}$$

We now add these equations and obtain, on the assumption that the relation expressed by [A] exists,

$$C_0 F(0) + \sum_{i=1}^{n} F(a_i) = \sum_{i=1}^{n} \epsilon_{a_i}. \qquad [B]$$

No restriction has been imposed upon $f(x)$ except that it be integral and rational. We may then put

$$f(x) = a_0^{np} \frac{x^{p-1}}{(p-1)!} \prod_{i=1}^{n} (a_i - x)^p,$$

in which p is a prime number and larger than the largest of the coefficients C_0, a_0, a_n. But by substituting this polynomial for $f(x)$ in $[B]$ we are able to prove the following three lemmas:*

(1) ϵ_{a_i} *and hence, since* n *is constant,* $\sum_{i=1}^{n} \epsilon_{a_i}$ *approaches the limit zero when* p *increases indefinitely.*

For, ϵ_{a_1} can be written in the form:

$$\epsilon_{a_1} = [-a_1 e^{(1-\theta)a_1} a_0^n \prod_{i=1}^{n} (a_i - \theta a_1)] \cdot \frac{[a_0^n \theta a_1 \prod_{i=1}^{n} (a_i - \theta a_1)]^{p-1}}{(p-1)!}.$$

Each of these brackets remains finite, no matter what value θ assumes between 0 and 1. The second factor, therefore, approaches 0, when $p = \infty$. Hence ϵ_{a_1} and likewise $\epsilon_{a_2}, \ldots \epsilon_{a_n}$, each approach zero as p is indefinitely increased.

* These lemmas are modifications of similar ones used by Hurwitz, *l. c.*, p. 221.

(2) $\sum_{i=1}^{n} F(a_i)$ *is an integer, divisible by* p. To show this we compute separately the values of $f(a_1), f'(a_1), \ldots$ in the expression

$$F(a_1) = f(a_1) + f'(a_1) + \cdots$$

The first p of these polynomials vanish, since the functions $f(x), f'(x), \ldots f^{(p-1)}(x)$ each contain $(a_1 - x)$ as a factor. The coefficients of the remaining polynomials $f^{(p)}(x), f^{(p+1)}(x), \ldots$ are integral and divisible by pa_0^{np}. Hence $F(a_1)$, and likewise each of the functions $F(a_2), \ldots F(a_n)$, contains the factor pa_0^{np}. The expression $\sum_{i=1}^{n} F(a_i)$ is a symmetric function of the roots $a_1, a_2, \ldots a_n$ of equation (1), and is of the degree $np-1$ in the a's. It may then, because of the factor a_0^{np}, be integrally expressed in terms of the integers $a_0, a_1 \ldots a_n$. That is, $\sum_{i=1}^{n} F(a_i)$ is an integer, and it is divisible by p.

(3) $C_0 F(0)$ *is an integer not divisible by* p. For, $f^{(p-1)}(x)$ is the first of the polynomials $f(x), f'(x), \ldots$ that does not vanish for $x = 0$; the coefficients of all of the subsequent polynomials, $f^{(p)}(x), f^{(p+1)}(x), \ldots$ are divisible by pa_0^{np}, and it follows as in the proof of (2) that the sum of these polynomials reduces, for $x = 0$, to an integer divisible by p. The value of $f^{(p-1)}(0)$ is $a_0^{np}(a_1 a_2 \ldots a_n)^p = (-1)^{np} a_0^{np-p} a_n^p$, so that

$$C_0 F(0) = (-1)^{np} C_0 a_0^{np-p} a_n^p + \text{an integer divisible by } p.$$

p was chosen a prime greater than C_0, a_0, a_n, and hence the first term is an integer not divisible by p.

From these three lemmas it appears that equation $[B]$ asserts that the sum of an integer not divisible by p and another integer divisible by p can be made small at will. This is absurd. The assumption that $[A]$ exists is therefore false and consequently π is transcendental.

THE UNIVERSITY OF NEBRASKA, FEBRUARY, 1900.

AN APPLICATION OF ELLIPTIC FUNCTIONS TO PEAUCELLIER'S LINK-WORK (INVERSOR).

By Arnold Emch.

In a recent article which appeared in the Annals of Mathematics,* I have studied a certain link-work and its relation with the elliptic integral of

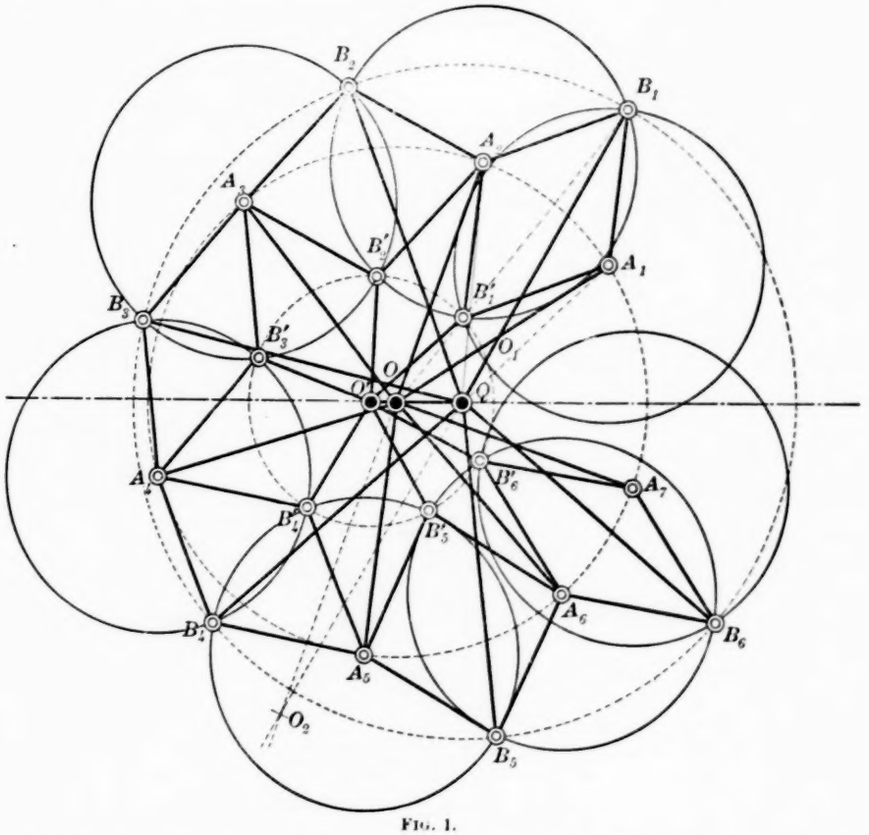

Fig. 1.

the first kind. By a geometrical transformation a class of circular series was

* "Illustration of the Elliptic Integral of the First Kind by a Certain Link-Work," ser. 2, vol. 1 (1900), p. 81.

obtained which led to *Poncelet's poristic polygons*, and suggested their equivalence with *Steiner's circular series*. By a slight generalization the link-work may be given such a form that, implicitly, it also includes Steiner's series. In this general form the element of the link-work is represented by *Peaucellier's Cell or Inversor*.*

1. The Link-Work. Assume in Fig. 1, the links $OA_1 = OA_2 = r_1$, $A_1B_1 = B_1A_2 = A_2B_1' = B_1'A_1 = r_2$, $QB_1 = R$, and $OQ = e$. The points O and Q are fixed, while all the others are movable. The arrangement of this link-work is evidently that of *Peaucellier's Inversor*. During the motion B_1 describes a circle having Q as a centre and QB_1 as a radius. Now, according to the properties of Peaucellier's Inversor $OB_1 \cdot OB_1' = r_1^2 - r_2^2 = \text{constant}$; consequently the point B_1' also describes a circle, which is inverse to the circle described by B_1 in an inversion having O as a centre and $\sqrt{r_1^2 - r_2^2}$ as a radius. Designating the variable distances QA_1 and QA_2 by ρ_1 and ρ_2, respectively, and putting $\rho_1^2 = x$, $\rho_2^2 = y$, $R + r_2 = a$, $r_2 + e = b$, $r_2 - e = c$, $R - r_2 = d$; furthermore designating by u and v the elliptic integrals

$$u = \int^x \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}}, \tag{1}$$

$$v = \int^y \frac{dy}{\sqrt{(y-a)(y-b)(y-c)(y-d)}}, \tag{2}$$

the fundamental relation

$$v - u = h \text{ (constant)} \tag{3}$$

may be found. The proof for this relation is essentially the same as the proof of the corresponding relation in my previous paper.† By inversion of the integrals (1) and (2) the elliptic functions $x = \lambda(u)$, $y = \lambda(v)$ are obtained. Thus the variables ρ_1 and ρ_2 may be expressed by elliptic functions, whose arguments have a constant difference. Here as there, other cells of equal size $(OA_2B_2A_3B_2' \cdot QB_2)$, $(OA_3B_3A_4B_3' \cdot QB_3)$, $\cdots$ may be added to the first, so as to form a general link-work which may close or may not close. Similar analysis to that employed in the former case leads to the corresponding result:

* A description of this particular link-work may be found in every modern text-book on kinematics. The constructions that it performs were proposed by Peaucellier in the *Nouvelles Annales de Mathématiques*, ser. 2, vol. 3 (1864), p. 414, and the link-work was published by him in the same journal, ser. 2, vol. 12 (1873), p. 71.

† *l. c.*, p. 84.

If a link-work of the prescribed kind, based upon two fixed circles (centres O and Q, radii OA_1 and QB_1) closes and contains n cells, then every other link-work based upon the same two circles closes and contains n cells.

It is clear that the fundamental relation (3) also holds in the cases of limited and unlimited link-works. The previous theorem, however, only holds for a closed link-work. Without entering into a further discussion of the various special cases which may present themselves, and which were treated

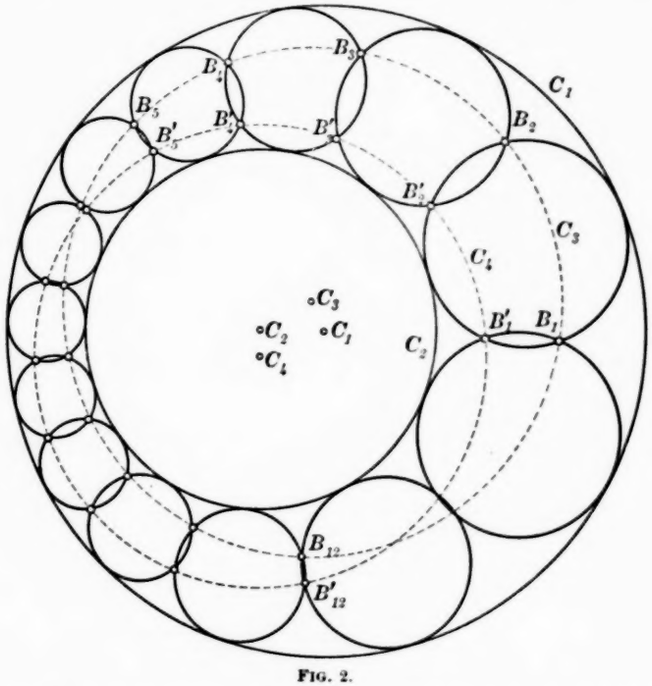

FIG. 2.

in the article referred to above, I shall now explain a geometrical transformation of the link-work which in its generality contains Steiner's and Poncelet's configurations.

2. Geometrical Transformation of the Link-Work. In Fig. 1, with $A_1, A_2, A_3, \ldots$ as centres and r_2 as a radius describe a series of circles. From the figure it is seen that the two circles having A_i and A_{i+1} as centres pass through the points B_i and B_i'. The properties of closing of these series are evidently the same as those of the link-work. All circles of the series are

tangent to two concentric circles. The figure can be generalized by an inversion and we have immediately the result:

*If each pair of consecutive circles of a series of circles, which all touch two fixed circles C_1 and C_2, intersect in two points B_i and B_i', and if the points B_1, B_2, B_3, . . . are situated on a circle C_3, then the points B_1', B_2', B_3', . . . are also situated on a circle C_4** (Fig. 2).

As the geometrical transformation of the link-work does not affect the properties of closing of the above series of circles we may state, in analogy with the link-work:

If a series of circles (generalized Steinerian series) of the prescribed kind, based upon two fixed circles C_1 and C_2 and a third circle C_3 (or C_4), closes and contains n circles, then every other series of circles based in the same manner upon the same three circles closes and contains n circles.

3. Steiner's Series and Poncelet's Polygons. By special disposition and by assuming some of the given circles as points or straight lines a great variety of circular series may be obtained. If the circles C_1, C_2 and C_3 are parts of a pencil of circles, then C_4 belongs to the same pencil. The series of circles obtained by this arrangement have been considered by Steiner.† One of the most interesting cases arises if one of the circles C_1 and C_2, for instance C_2, degenerates into point O. All circles of the series pass through O, and the circle C_4 coincides with O. Any inversion having O as a centre transforms all circles of the series into straight lines which are inscribed in a circle C_3' and circumscribed about a circle C_1', *i. e.* the limited portions of these lines form a polygon which is inscribed in one and circumscribed about another circle. The properties of closing of these polygons, which are called Poncelet's polygons, are the same as those of the general series of circles. Poncelet's constructions also result directly from a geometrical transformation of the link-work in which $r_1 = r_2$. This case was exclusively studied in the article to which I have referred in the beginning.

UNIVERSITY OF COLORADO, OCTOBER 1900.

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* This statement may be generalized in such a manner that instead of a circle C_3 any curve is assumed. From Fig. 1 it can easily be proved that in this case B_1', B_2', B_3', . . . are situated on a curve which is the inverse of the first with regard to the centre O. The special case above has been formulated in view of its subsequent application.

† *Werke*, vol. 1, pp. 19-76.

NOTE ON THE GEOMETRICAL TREATMENT OF CONICS.

By Charlotte Angas Scott.

1. It is strange that the treatment of the conic by means of projective flat pencils (or of projective ranges, when the conic is regarded as an envelope) should have so completely displaced the beautiful involution treatment adopted by von Staudt. This enabled him to deal with the points and tangents of the curve simultaneously and symmetrically, with the result that all polar properties are made almost intuitive. In comparison with the simplicity of this, the method of projective pencils is cumbrous; and in comparison with the breadth of von Staudt's treatment, the other is positively niggling. A short account of the development of the theory of conics by von Staudt's method (with a few simplifications) may be of interest.

2. Von Staudt defines involution in a primitive form of the first grade (range of points, flat pencil, or axial pencil) by means of two projective forms on the same base, remarking immediately that this is really a projective pairing of the elements. He might with advantage have employed this, the more characteristic property of the involution, as the defining property: a primitive form of the first grade is arranged in involution if its elements are associated in pairs in such a manner that any figure is projective with its conjugate. This relation holds, for example, if the associated elements are harmonic with reference to a fixed pair of elements. With this definition, we have the usual proofs and properties, namely, that two pairs of elements determine the involution, that the correspondent to any arbitrary element can be determined by a linear construction, that there may be two self-corresponding elements (double elements) but not more, and that these are harmonic with respect to any pair in the involution.

Taking in the next place a primitive form of the second grade, that is, either a plane with all its points and lines, or a point with all the planes and lines through it, it is at once evident that we have to consider the possibility of pairing the elements projectively in two essentially distinct ways; for we may wish to associate elements of the same kind, or elements different in kind. The first does not give any very important results; we consider the second only, and, for definiteness, only as applied to a plane.

3. Supposing such a projective pairing of the elements to have been accomplished, then to a line p corresponds a point P, and *vice versa*; to any figure composed of points $PQR\ldots$ and lines $uvw\ldots$ there corresponds a figure composed of lines $pqr\ldots$ and points $UVW\ldots$; to the line AB corresponds the point ab; to collinear points $PQR\ldots$, concurrent lines $pqr\ldots$, the range and pencil being moreover projective. This relation of a point and line is referred to as polar, the point being the pole of the line, the line the polar of the point; or, we may say, each being polar to the other. The plane thus arranged is a polar system. If the polar of P, namely p, passes through Q, then P lies on q, that is, the polar of Q passes through P; the points P, Q are said to be conjugate, as are also the lines p, q. The point R, common to p, q, has for its polar the line PQ; the triangle PQR has its sides and vertices associated, it is a self-polar or self-conjugate triangle.

Let $A,B,C,\ldots$ be points on any line, which is met by the polars $a,b,c,\ldots$ in $A',B',C',\ldots$; then $AA', BB', \ldots$ are conjugate. If we start with A', we obtain A, etc; thus the points of the line are paired; and as

$$(AA'BB'CC'\ldots) = (aa'bb'cc'\ldots) = (A'AB'BC'C\ldots),$$

they are projectively paired. Hence pairs of conjugate points on any line form an involution; and pairs of conjugate lines through any point form an involution.

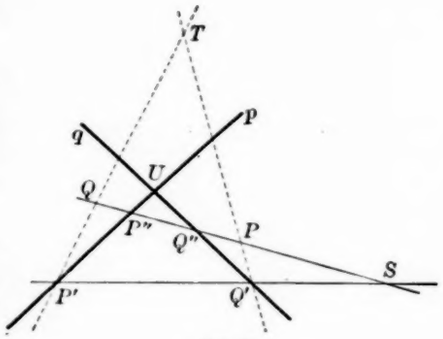


FIG. 1.

4. Let PP', QQ' be any two pairs of conjugates that are not collinear, and let $PQ, P'Q'$ meet in S, PQ', $P'Q$ in T; then S, T are conjugate. (Hesse's theorem.)

For if p, q (fig. 1) meet SPQ in P'', Q'', then PP'', QQ'' are two pairs of conjugate points on the line SPQ, hence they determine the involution of conjugate pairs on this line, and consequently they determine the correspondent to S. This is at once obtained by means of the complete quadrangle $P'Q'TU$; the line TU determines S''. But as S is on PQ, the polar of S goes through $p\,q$, that is, through U. Hence the polar of S is $S''U$, that is, TU; which shows that S and T are conjugate points.

5. This projective pairing of unlike elements of the plane can be accomplished; for a polar system of this nature is determined if we associate the vertices and opposite sides of some one triangle, and then associate any one point of the plane with any one line, these being chosen arbitrarily, and of course not incident on any element of the triangle. To prove this, let ABC be the triangle, Pp the arbitrary pair; let AP and p meet BC in D, D', then D' is associated with AD, and DD' are therefore conjugate points (fig. 2). Thus on the line a two pairs in the involution of conjugate points are known, and the correspondent to any other point is therefore linearly determinable. Hence to construct the line q, associated with any arbitrary point Q, let AQ, BQ, CQ, meet a, b, c in X, Y, Z; then the line q is determined by any two of the conjugates to X, Y, Z, on a, b, c, namely, X', Y', Z'.

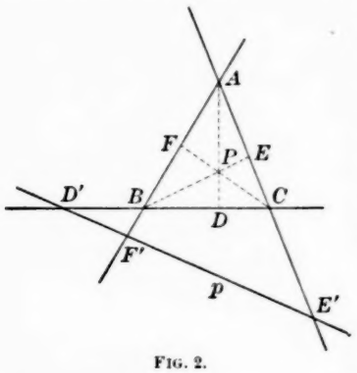


FIG. 2.

That the three points X', Y', Z' are actually collinear, or, what is equivalent, that three collinear points X', Y', Z' satisfy the relations of involution on a, b, c, appears as follows. We have on b, c, the involutions

$$(AC,\ EE',\ YY'),\ (AB,\ FF',\ ZZ').$$

Let BE, CF, meet in P, BY, CZ in Q (fig. 3); then AP, AQ meet a in D, X. Let $E'F'$, $Y'Z'$ intersect in S, and let them meet a in D', X', and the line PQ in P', Q'. Let PQ meet a, b, c in A', B', C', and the lines SA, SB, SC in A'', B'', C''. In virtue of the involution on b,

$$(ACEY) = (CAE'Y');$$

projecting the left hand side of this from B, and the right hand side from S, on to PQ, this gives

$$(C'A'PQ) = (C''A''P'Q').$$

Similarly

$$(B'A'PQ) = (B''A''P'Q').$$

Combining these,

$$(A'B'C'PQ) = (A''B''C''P'Q'),$$

and therefore

$$(B'C'PQ) = (B''C''P'Q').$$

Projecting the left hand side of this from A, and the right hand side from S, on to a, it gives

$$(CBDX) = (BCD'X'),$$

which proves that (BC, DD', XX') is an involution.

6. Consider the involution of conjugate points on any line; if this has

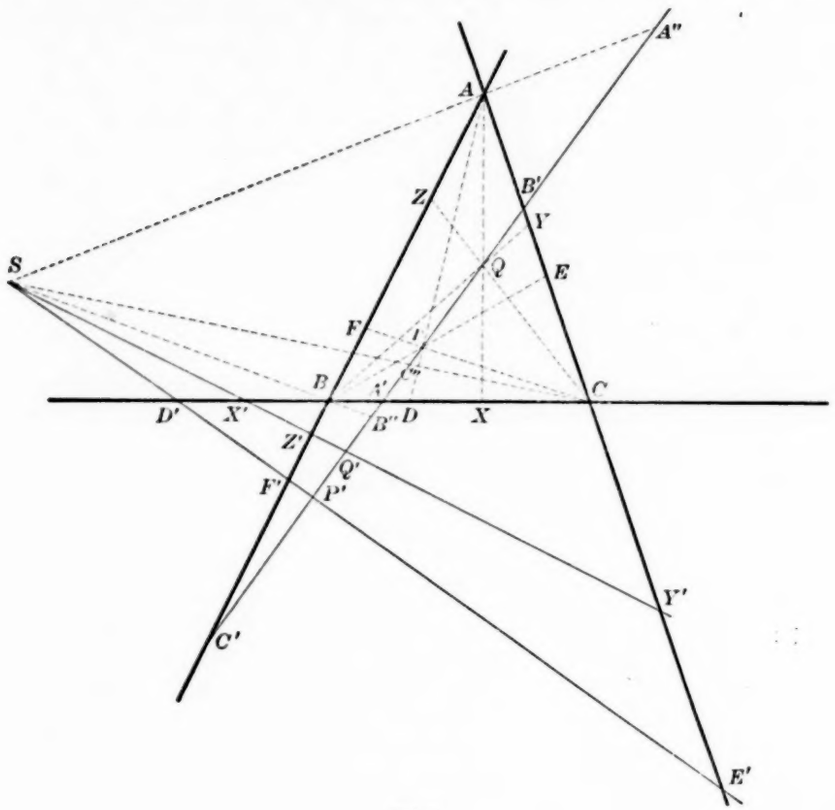


FIG. 3.

double points, each is a point that lies on its own polar, it is conjugate to itself in the plane, and conversely. Similarly all lines that pass through their own poles are the double lines of involution pencils of conjugate lines. Supposing imaginary elements to have been introduced before this stage in the work is reached, a plan which seems highly desirable, the result is that on an

arbitrary line there are two self-conjugate points, real or imaginary, and through an arbitrary point there are two self-conjugate lines, real or imaginary.

Let P, p be one such point and line, real or imaginary. First consider any point T on p. The polar of T passes through P, and is distinct from p, hence it does not pass through T. That is to say, no point on p, other than P, lies on its own polar; on the line p there is only the one self-conjugate point P, and to this every point on the line p is conjugate.

In the next place, consider the line t, that is, any line through P. On this there is an involution of conjugate points, which has one double point P, real or imaginary, and consequently another Q, of the same nature. Hence on every line through P, other than p, there is one self-conjugate point distinct from P. There is thus a system of these points, $P\,Q\,R\,\ldots$, in perspective with (and consequently in (1, 1) correspondence with) the pencil of lines through any one of them, P. This system is such that any line through one of its points passes through precisely one other, with the exception of one line at every point. These excepted lines, $p\,q\,r\,\ldots$, form a system in perspective with (and consequently in (1, 1) correspondence with) the range determined on any one of them; the system is such that through any point on one of its lines, p, another line passes, unless the point is P, the pole of p. The point-system $P\,Q\,R\,\ldots$ and the line-system $p\,q\,r\,\ldots$ consequently indicate one and the same curve, which regarded as a locus is of the second order, and regarded as an envelope is of the second class. This curve is called a conic; it has points, P, and tangents, p, meeting the curve at P only.

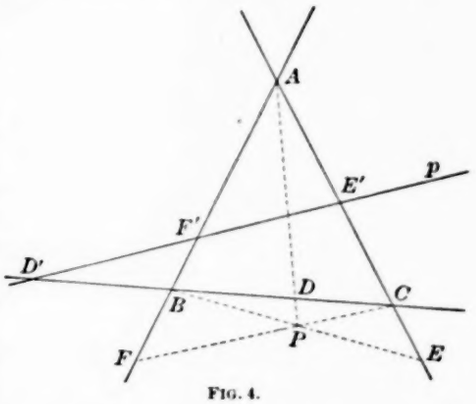


FIG. 4.

7. It may happen that a real polar system gives rise to an imaginary conic (von Staudt, *Beiträge zur Geometrie der Lage*, p. 107). If a point P and its polar p are so placed that it is impossible to pass from P to p without crossing a side of the triangle ABC (or, we may say, if P and p are separated), then the involution of conjugate points on p has imaginary double ele-

ments. The double points on p are determined by the double lines of the involution $P(DD', EE', FF')$; fig. 2 shows one of the two possible arrangements of P, p, and as PE, PE' are necessarily separated by PF, PF', the double lines are imaginary. Similarly if P be outside the triangle, a case which is shown in fig. 4. Now if one point P is separated from its polar p, this is true for every other pair Qq. For referring to fig. 3, XX' belong to the involution determined by BC, DD', hence if D, D' are separated by B, C, this is true of X, X' also, and similarly of Y, Y' and Z, Z'. Thus if P, p are chosen so as to be separated by the triangle ABC, every line meets the conic in imaginary points, hence the conic is entirely imaginary; while if P, p are not separated by the triangle ABC, two sides of the triangle meet the conic in real points; the conic is then real, being met by every line through one of these real points in another real point.

The case of an imaginary conic through two or four real points does not present itself in connection with a real polar system; it arises in the consideration of imaginary polar systems.

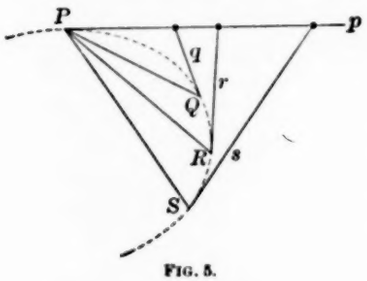


FIG. 5.

8. Since the tangents at P, Q, are the polars of P, Q, they meet in the pole of PQ. Thus the pole of a line is the point of intersection of the tangents at the points where the line cuts the conic; and the polar of a point is the chord of contact of tangents from the point.

The tangents at $Q, R, S, \ldots$ meet the tangent at P, that is, p, in the poles of $PQ, PR, PS, \ldots$; hence the projective pairing of the elements of the plane shows that the range determined on p by the tangents at $Q, R, S, \ldots$ is projective with the pencil determined at P by the points $Q, R, S, \ldots$ (fig. 5).

On any line of the plane there is an involution of conjugate points, whose double points are the points where the line meets the conic. Let U, u be a pole and polar, and let any line through U cut the conic in P, Q, and u in U'. Then as U, U', being conjugate points, are harmonic with respect to P, Q, by varying the line through U we obtain the theorem: — the locus of a point harmonically separated from U by the conic is a straight line u, the polar of U. Thus the polar relation of a point and line in the plane system is exactly equivalent to the ordinary polar relation with respect to the conic.

9. It has been shown that the points of a conic subtend at any one point, P, a pencil that is projective with the range determined on p by the tangents. It has now to be shown that the pencils determined at any two points are projective. The proof given by von Staudt is unnecessarily complicated and lengthy; a shorter one is here given, which is really a proof of Chasles' theorem, so simple that it is difficult to believe that it has not been given before.

Let W be the pole of PQ (fig. 6); let R be any point on the conic, and let PR, QR meet QW, PW, in Q', P'; let r meet q, p in U, V. Then since

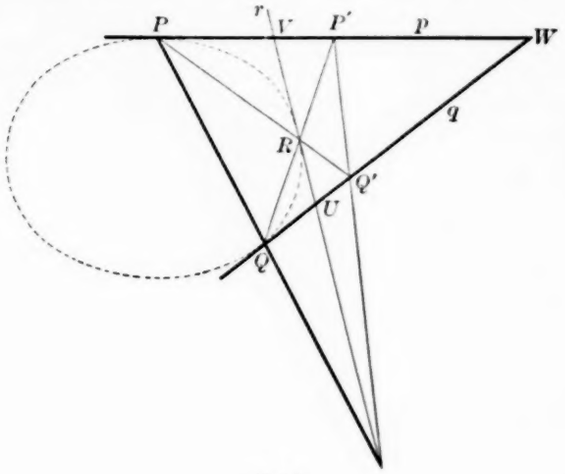


FIG. 6.

Q' lies on the polar of Q and on the polar of V, VQ is the polar of Q'; hence VQ, VQ' are conjugate lines, and are therefore harmonically separated by the tangents from V, that is, by VP, VR. Thus (QQ', UW) is harmonic. Hence since Q, W are fixed points, the ranges described on QW by Q' and U are projective;* that is, symbolically,

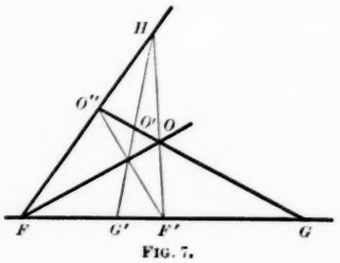


FIG. 7.

LEMMA. If (FF', GG') is harmonic, where F, G are fixed, F', G' variable, then F', G' describe projective ranges. For in the figure by which the harmonic set is determined (fig. 7), we can keep the lines FO, FO'', GO, and the point O', fixed; we have then a series of pencils in perspective,

$$O(F' \ldots) = O(H \ldots) = O'(H \ldots) = O'(G' \ldots);$$

hence

$$\text{range } (F' \ldots) = \text{range } (G' \ldots).$$

$$(Q' \ldots .) = (U \ldots .).$$

Similarly $$(P' \ldots .) = (V \ldots .).$$

But since V is the pole of PR,

$$\text{range } (V \ldots .) = \text{pencil } P(R \ldots .)$$
$$= \text{range } (Q' \ldots .).$$

Hence the ranges $(U \ldots .)$, $(V \ldots .)$ are projective with one another, and with the pencils $P(R \ldots .)$, $Q(R \ldots .)$, which proves not only Steiner's theorem, that the points of a conic are the intersections of corresponding rays of projective pencils, and the tangents to the conic the lines joining corresponding points of projective ranges, but also Chasles' theorem, namely, that four given points on a conic determine at any point of the curve a pencil projective with the range determined on any tangent to the curve by the tangents at the given points.

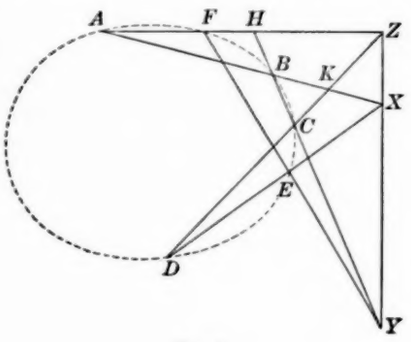


FIG. 8.

10. Von Staudt's own proof of Pascal's theorem (or of Brianchon's theorem) is simple enough. Since the pencils $D(BCEA)$, $F(BCEA)$ are projective (fig. 8), the ranges in which they are cut by any two transversals are projective. Using the transversals BA, BC, this gives

$$(BKXA) = (BCYH),$$

and as these two projective ranges have B in common, they are in perspective. Hence KC, XY, AH are concurrent, that is, CD and AF meet on the line XY.

His proof of Desargues' theorem is unnecessarily long; a simpler one is given by later writers, *e. g.* by Reye.

Let a conic through $ABCD$ cut the transversal XY in P, P'. Then (fig. 9)

$$D(ACPP') = B(ACPP'),$$

therefore

$$(Y'X'PP') = (XYPP') = (YXP'P),$$

which shows that PP' are a pair in the involution XX', YY'. Hence all

conics through $ABCD$, including the three line-pairs, cut any transversal in involution.

All the purely projective properties of conics can be readily deduced from these few theorems.

11. A comparison with the treatment of conics based on projective pen-

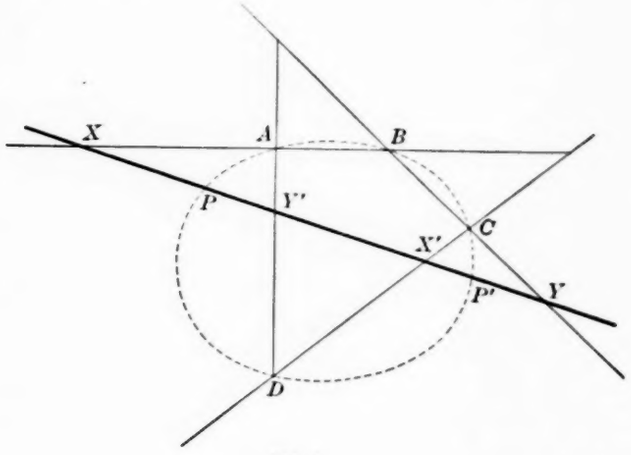


FIG. 9.

cils or ranges, as given for example in Reye's *Geometrie der Lage*, or in the article "Geometry" (Section II) in the *Encyclopædia Britannica*, will prove the great superiority of von Staudt's method, which is exhibited specially in everything that relates to poles and polars with reference to the conic.

BRYN MAWR COLLEGE, PENNSYLVANIA, APRIL 1900.

ON TWO-DIMENSIONAL FLUID MOTION THROUGH SPOUTS COMPOSED OF TWO PLANE WALLS.

By R. A. Harris.

It is well known that the functions $\sin z$, $\cos z$, $\sinh z$, $\cosh z$, transform lines parallel to the axes of x and y, where $z = x + iy$, into systems of confocal ellipses and hyperbolas; also, that if we take the hyperbolas as stream lines, the two-dimensional motion represented by these systems is that of a boundless liquid sheet of uniform thickness flowing through an aperture two units wide made in a thin plane partition or wall extending to infinity in either direction. Helmholtz* discovered the curve systems for the case in which infinite plane walls extend from the edges of the aperture or orifice, and perpendicular to its plane, to infinity in one and the same direction, thus forming the parallel sides of a spout. This paper is concerned with a transformation, equation (1) below, which comprises these two cases together with all intermediate ones.

Assume the equation

$$Z = X + iY = \frac{\epsilon^{\epsilon}(1-\epsilon)^{1-\epsilon}}{\sin \epsilon\pi}\left[e^{(1-\epsilon)z} - e^{-\epsilon z}\right], \tag{1}$$

where ϵ is any real positive quantity ranging from $\frac{1}{2}$ to 0, and $z = x + iy$. The motion is supposed to take place in the Z-plane. The constant factor is, as will soon appear, such that the width of the orifice shall be 2 units in all cases. From equation (1)

$$X = \left(\frac{\epsilon}{1-\epsilon}\right)^{\epsilon}\frac{1-\epsilon}{\sin \epsilon\pi}\left[e^{(1-\epsilon)x}\cos(1-\epsilon)y - e^{-\epsilon x}\cos \epsilon y\right], \tag{2}$$

$$Y = \left(\frac{\epsilon}{1-\epsilon}\right)^{\epsilon}\frac{1-\epsilon}{\sin \epsilon\pi}\left[e^{(1-\epsilon)x}\sin(1-\epsilon)y + e^{-\epsilon x}\sin \epsilon y\right]. \tag{3}$$

* *Berliner Sitzungsberichte*, April 23, 1868, p. 224 or *Wissenschaftliche Abhandlungen*, vol. 1, p. 154. A drawing is given by Maxwell, *Electricity and Magnetism*, vol. 1, fig. 12; and by Lamb, *Hydrodynamics*, p. 83.

If $y = \pi$, we have

$$X = -\left(\frac{\epsilon}{1-\epsilon}\right)^{\epsilon} \frac{1-\epsilon}{\tan \epsilon\pi} \left[e^{(1-\epsilon)x} + e^{-\epsilon x}\right], \tag{4}$$

$$Y = \left(\frac{\epsilon}{1-\epsilon}\right)^{\epsilon} (1-\epsilon) \left[e^{(1-\epsilon)x} + e^{-\epsilon x}\right], \tag{5}$$

and so

$$\frac{Y}{X} = -\tan \epsilon\pi; \tag{6}$$

that is, the line in the Z-plane corresponding to $y = \pi$ is a straight line making an angle of $-\epsilon\pi$ with the X-direction. Expressions (4) and (5) show that X is for this line always negative and Y positive. With $y = \pi$, X will be a maximum and Y a minimum when x satisfies the equation

$$e^{x} = \frac{\epsilon}{1-\epsilon}. \tag{7}$$

This substituted in (4) and (5) gives

$$X = -\frac{1}{\tan \epsilon\pi}, \quad Y = 1 \tag{8}$$

for the coordinates of the near end of the wall which extends thence to infinity toward the left and upward.

By referring back to (2) and (3) we see that if y were replaced by $-y$, X would remain as before while Y would have its sign changed. In particular, the line $y = -\pi$ will give a straight line in the Z-plane extending from the point $(-\cot \epsilon\pi, -1)$ to infinity toward the left and downward, making an angle of $\epsilon\pi$ with the X-direction. These two lines in the Z-plane are the traces of two plane walls which form a spout with an orifice 2 units in width.*

* The function Z, regarded as a function of z, is single valued and analytic throughout the interior of the strip of the z-plane bounded by the lines $y = \pi$, $y = -\pi$; and this function maps the boundary of the strip on the lines of the Z-plane above considered. If we distinguish between the two sides of these lines, regarding the lines as doubly counting, then the boundary of the strip goes over in a one-to-one manner and continuously into the two edges of the slit made by cutting the Z-plane along the two lines in question. That the *interior* of the strip goes over conformally into the *whole Z-plane* exclusive of the cut, follows from an easy generalization of the theorem usually applied in problems of this sort. *Cf.* Picard, *Traité d' Analyse*, vol. 2, p. 280. [Ed.]

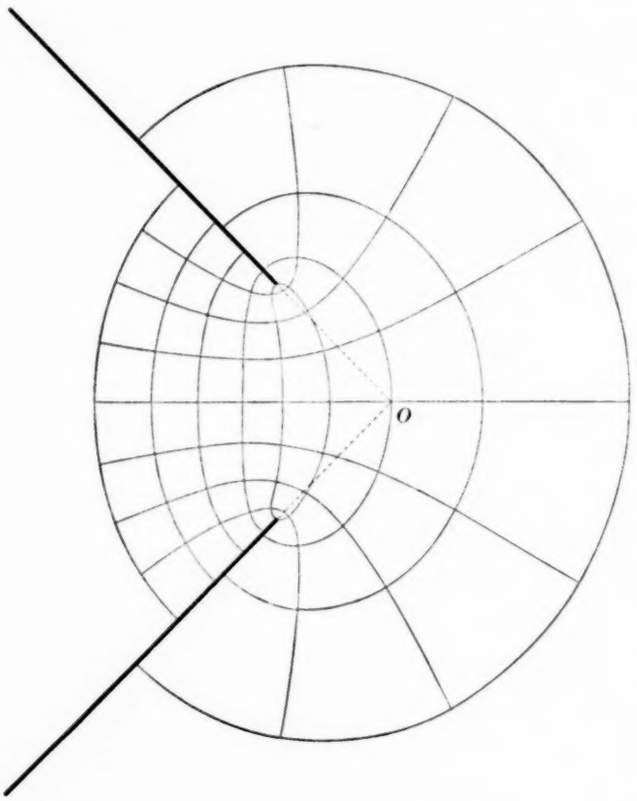
If we make $y = \beta$ and allow x to increase indefinitely, it follows from (2) and (3) that

$$\lim_{x = \infty} \frac{Y}{X} = \tan (1 - \epsilon)\beta, \tag{9}$$

and similarly

$$\lim_{x = -\infty} \frac{Y}{X} = -\tan \epsilon\beta. \tag{10}$$

It thus appears that the stream line which corresponds to $y = \beta$ makes an angle of nearly $-\epsilon\beta$ with the X-direction when extended sufficiently far between the



walls, and an angle of nearly $(1 - \epsilon)\beta$ when extended sufficiently far outside of them.

Special Cases. If $\epsilon = \frac{1}{2}$, then (1), (2), and (3) become $Z = \sinh \frac{1}{2} z$, $X = \sinh \frac{1}{2} x \cos \frac{1}{2} y$, $Y = \cosh \frac{1}{2} x \sin \frac{1}{2} y$. $x = \alpha$ and $y = \beta$ transform into systems of confocal ellipses and hyperbolas whose foci are at the points $(0, \pm 1)$. The dihedral angle between the plane walls is 180°.

If $\epsilon = \frac{1}{4}$, we have the case shown in the accompanying figure. The walls make angles of ±45° with the X-direction, or an angle of 90° with each other. Equations (2) and (3) become

$$X = 0.806 \left[e^{\frac{1}{4}x} \cos \tfrac{3}{4} y - e^{-\frac{1}{4}x} \cos \tfrac{1}{4} y \right], \tag{12}$$

$$Y = 0.806 \left[e^{\frac{1}{4}x} \sin \tfrac{3}{4} y + e^{-\frac{1}{4}x} \sin \tfrac{1}{4} y \right]. \tag{13}$$

The values assigned to α range from $-\frac{3}{4}\pi$ to $\frac{1}{2}\pi$, and those assigned to β, from $-\pi$ to π. The equipotential line connecting the two foci, and setting out from them in the direction of the walls, is the line $e^x = \dfrac{\epsilon}{1-\epsilon}$ when transformed to the Z-plane; it has not been drawn.

As ϵ becomes very small, the walls make a very small angle with the X-direction, and so, if produced, their intersection becomes removed too far to the right of the opening for a convenient X-origin. When ϵ approaches 0, the origin recedes indefinitely and we have in the limit the case given by Helmholtz. We can take the origin at the mouth of the spout if we introduce new variables z_1 and Z' such that

$$z_1 = z - \log \frac{\epsilon}{1-\epsilon}, \qquad Z' = Z + \cot \epsilon\pi \tag{14}$$

If we now allow ϵ to approach 0 as its limit Z' approaches Helmholtz's function, Z_1, and we have in the limit:

$$Z_1 = \frac{1}{\pi}\left(e^{z_1} + z_1 \right) + \frac{1}{\pi}. \tag{15}$$

WASHINGTON, D. C., FEBRUARY, 1900.

ON A SPECIAL CLASS OF ABELIAN GROUPS.

By G. A. Miller.

1. Let $s_1, s_2, \ldots, s_{g_1}$ represent the $\phi(g)$* natural numbers which are less than g and prime to it. These g_1 numbers constitute a group (G_1) with respect to multiplication if the smallest positive residues mod g are taken in place of the products; for, from the relation $s_i s_h = qg + r$, $0 < r < g$, it follows that, if g and r had a common divisor, a prime factor of this divisor would have to divide either s_i or s_h, and thus these numbers would not both be prime to g. The other necessary conditions for a group are evidently satisfied. Such groups have recently received considerable attention, and, in the second edition of his Algebra, Weber† has called them the most important example of abelian groups of finite order. The main objects of this paper are to show that all these groups are groups of isomorphisms of the cyclical groups and to study some of their special properties.

2. By the *group of isomorphisms* of a given group G' whose elements (operators) are $t_1, t_2, \ldots, t_{g'}$ is meant the following group. Let G' be simply isomorphic with itself, the element t_i corresponding to $t_{\omega(i)}$, and form the substitution

$$J = \begin{pmatrix} t_1 & t_2 & \ldots & t_{(g')} \\ t_{\omega(1)} & t_{\omega(2)} & \ldots & t_{\omega(g')} \end{pmatrix}$$

Then the totality of such substitutions constitutes a group‡ and this is the group in question. The following considerations lead to the group of isomorphisms of a group generated by a single element S. Let T be any element that satisfies the equation $T^{-1}ST = S^\alpha$. Raising both members of this equation to the βth power we have $T^{-1}S^\beta T = S^{\alpha\beta}$. We may express this well known result in words as follows: If an element transforms a generator of a cyclical group

* *Cf.* Chrystal's *Algebra*, vol. 2, 1889, p. 511.

† Weber, *Lehrbuch der Algebra*, 2d ed., vol. 2, 1899, p. 60.

‡ Hölder, *Math. Annalen*, vol. 43 (1893), p. 314. *Cf.* Moore, *Bull. Amer. Math. Soc.* vol. 1 (1894), p. 61.

into a given power. it transforms each one of the elements of this cyclical group into the same power.

In any holomorphism of a cyclical group G of order g with itself the $\phi(g)$ elements of order g must correspond to each other. A given generator S must therefore correspond to S^k, where k is prime to g. Since any holomorphism of a group with itself can be obtained by transforming all the elements of the group by a single element* it follows from the preceding paragraph that such a holomorphism is obtained by making each element correspond to its kth power. This result follows also directly from the fact that in any holomorphism like powers of corresponding elements yield corresponding elements. Since k is prime to g we may assume that $k = s_a$ $(0 < a \leqq g_1)$ and that the $s_1, s_2, \ldots s_{g_1}$ powers of S correspond to the $s_a s_1, s_a s_2, \ldots, s_a s_{g_1}$ powers respectively in the given holomorphism of G with itself. Hence any holomorphism of G with itself corresponds to a substitution of the form

$$J_a = \begin{pmatrix} s_1, & s_2, & \ldots & s_{g_1} \\ s_a s_1, & s_a s_2, & \ldots & s_a s_{g_1} \end{pmatrix} \mod g$$

and for every substitution of this form there is a holomorphism of G with itself. That is, the totality of these substitutions constitutes the group of isomorphisms of G. This group is obviously holomorphic with the group $s_1, s_2, \ldots, s_{g_1}$.

Thus the first of the two objects mentioned above is attained, viz. it is proved that the group G_1, whose elements are the $\phi(g)$ numbers that are less than g and prime to it, is the group of isomorphisms of the cyclical group of order g. This connects explicitly Burnside's study of the groups of isomorphisms of cyclical groups† and Weber's treatment of such groups as G_1.‡ In what follows we aim to establish a few properties of these groups which do not seem to have been explicitly noticed.

3. While the group of isomorphisms of a cyclical group is always abelian we shall prove that there are many abelian groups which cannot be the groups of isomorphisms of cyclical groups. Hence the groups which can be represented in the same manner as G_1 is represented above constitute a special class of abelian groups. We proceed to consider some of the conditions that must be

* Frobenius, *Berliner Sitzungsberichte*, 1895, p. 184.

† Burnside, *Theory of Groups of a Finite Order*, 1897, p. 240.

‡ Weber, *l. c.*

satisfied in order that an abelian group (A) may belong to this class. When (A) is cyclical the matter is quite simple. It is necessary and sufficient that its order (a) is the exponent to which a primitive root of some number belongs. That is, whenever $a = p^{\alpha}(p-1)$*, p being an odd prime number and α being any positive integer including 0, the cyclical group (A) belongs to the given class, and only then. The lowest two even numbers which are not of the form $p^{\alpha}(p-1)$ are 8 and 14; hence these numbers are the lowest orders of cyclical groups of an even order that cannot be the groups of isomorphisms of any groups whatever.

If g is written in the form: $g = 2^{\alpha_0}p_1^{\alpha_1}p_2^{\alpha_2}\ldots$ ($p_1, p_2 \ldots$ being different odd prime numbers) G is the direct product of its subgroups of orders $2^{\alpha_0}, p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots$ and G_1 is the direct product of the groups of isomorphisms of these subgroups.† Since the group of isomorphisms of a cyclical group whose order is a power of an odd prime number is cyclical, it follows from the above that G_1 is the direct product of the cyclical groups of orders $p_1^{\alpha_1-1}(p_1-1)$, $p_2^{\alpha_2-1}(p_2-1)\ldots$ when $\alpha_0 = 0$ or 1. When $\alpha_0 > 1$ we have to add a group of order 2 and a cyclical group of order 2^{α_0-2} to these factor groups in order to obtain G_1.‡

Since G_1 is the direct product of groups of even orders, g_1 *is always even*. It can clearly be any even number of the form $2^{\beta_0}p_2^{\beta_1}p_2^{\beta_2}\ldots.(p_1-1)(p_2-1)\ldots$ The smallest two natural numbers which are not of this form are 14 and 26§; hence these numbers are the lowest even orders of groups that cannot be groups of isomorphisms of any cyclical groups whatever. It is evident that the highest prime factor of g_1 cannot exceed the highest prime factor of g.

In conclusion I consider briefly the groups G_1 of order g_1, the power of a prime; this prime is then necessarily 2. Therefore g has the form: $g = 2^{\alpha_0}p_1p_2\ldots$, where the odd primes (if any) are of the respective forms: $p_1 = 2^{\beta_1}+1$, $p_2 = 2^{\beta_2}+1, \ldots; \beta_1 < \beta_2 < \ldots$. The group G_1 is then the direct product of cyclic groups of orders $2^{\beta_1}, 2^{\beta_2}, \ldots$, and further, if $\alpha_0 > 1$, of cyclic groups of orders 2 and 2^{α_0-2}. These orders are the invariants of the abelian group G_1. They are all distinct except in the cases: (1°) $\beta_1 = 1$, $\alpha_0 > 1$; (2°) $\beta_1 > 1$, $\beta_\gamma = \alpha_0 - 2 > 1$; (3°) $\alpha_0 - 2 = 1$. The necessary and sufficient con-

* *Cf.* Dirichlet-Dedekind, *Zahlentheorie*, 1879, p. 340.

† *Bull. Amer. Math. Soc.*, vol. 5 (1899), p. 296.

‡ *Cf.* Burnside, *l. c.*

§ Lucas, *Théorie des nombres*, 1891, p. 394.

dition that G_1 contains three equal invariants is that (1°) and (3°) are both satisfied: hence G_1 cannot contain three equal invariants unless each is two. When $\beta_1 = 1$ and $\beta_\gamma = a_0 - 2 > 1$, G_1 contains two pairs of equal invariants. These are the only cases in which G_1 can contain more than one pair of equal invariants.

CORNELL UNIVERSITY, APRIL 1900.

THE THEORY OF LINEAR DEPENDENCE.

By Maxime Bôcher.

The subject treated in the following pages is not only one which, owing to its numerous applications, is of considerable importance, but, while distinctly elementary, illustrates well one of the most striking tendencies of modern algebraic and analytic work, namely, the tendency not to be satisfied with results which are merely true "in general," *i. e.* with more or less numerous exceptions, but to strive for theorems which are *always* true.* For these reasons, and because, so far as I know, no adequate treatment of the subject as a whole exists, I venture to offer the following presentation to the readers of the Annals.

I wish to call special attention to the application made in §2 to the theory of linear equations. The more usual course is to proceed in the reverse order and to deduce the theory of linear dependence from the theory of linear equations. I think the order here adopted has the advantage of separating from one another with more distinctness the various steps in the reasoning.

1. Definitions and General Theorems. Geometrical Illustrations. Linear dependence may be regarded as a generalization of the conception of proportionality. For this purpose we will lay down the following definition of proportionality, which is at once seen to be equivalent to the ordinary definition:

The two sets of constants

$$x_1', \; x_2', \; \ldots \; x_n',$$
$$x_1'', \; x_2'', \; \ldots \; x_n'',$$

are said to be proportional to each other if two constants c_1 *and* c_2, *not both zero, exist such that*:

$$c_1 x_i' + c_2 x_i'' = 0 \qquad (i = 1, 2, \ldots n).$$

Instead of two sets of constants we now consider m sets, and give the following

*The great importance of this tendency will be apparent if we remember that when we apply a theorem, it is usually to a special case. If we know merely that the theorem is true "in general," we must first consider whether the special case to which we wish to apply it, is not one of the exceptional cases where the theorem fails.

DEFINITION. *The m sets of n constants each:*

$$x_1^{[i]}, x_2^{[i]}, \ldots x_n^{[i]} \qquad (i = 1, 2, \ldots m)$$

are said to be linearly dependent if m constants $c_1, c_2, \ldots c_m$, not all zero, exist such that:

$$c_1 x_j' + c_2 x_j'' + \ldots + c_m x_j^{[m]} = 0 \qquad (j = 1, 2, \ldots n).$$

If this is not the case, the sets of quantities are said to be linearly independent.

In the same way we generalize the familiar conception of the proportionality of two functions as follows:

DEFINITION. *The m functions (of any number of independent variables) $f_1, f_2, \ldots f_m$ are said to be linearly dependent if m constants $c_1, c_2, \ldots c_m$, not all zero, exist such that:*

$$c_1 f_1 + c_2 f_2 + \ldots + c_m f_m \equiv 0.^*$$

If this is not the case the functions are said to be linearly independent.†

The following theorems about linear dependence, while almost self evident, are of sufficient importance to deserve explicit statement.

I. *If m sets of constants (or if m functions) are linearly dependent, it is always possible to express one—but not necessarily any one—of them linearly in terms of the others.*

This is seen at once if we remember that at least one of the c's is not zero. The relations (or relation) in which the c's occur can then be divided through by this c.

II. *If there exist among the sets of constants (or among the functions) a smaller number of sets (or of functions) which are linearly dependent, then the m sets (or the m functions) are linearly dependent.*

III. *If any one of the m sets of constants consists exclusively of zeros (or if any one of the m functions is identically zero), the sets (or the functions) are linearly dependent.*

In order to show at once that this conception of linear dependence is important, we will give here a few simple geometrical applications, leaving the reader to supply the proofs of the statements made.

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* We use the symbol $\equiv$ here and in what follows (except in §5 (5) where, as is obvious from the context, it denotes a congruence) to denote an identity, *i. e.* an equality which holds either for all values of the independent variables, or at least for all the values with which we have to deal.

† We might clearly go farther and consider the linear dependence of m sets of n functions each. The two cases of the text would be merely special cases from this general point of view.

The sets of n constants with which we had to deal in the first definition may advantageously be regarded as the homogeneous coordinates of points in space of $n-1$ dimensions. It will then be convenient to speak of the linear dependence or independence of these points. The geometrical meaning of linear dependence here, will be at once evident from the following theorems for the case $n = 4$:

Two points are linearly dependent when, and only when, they coincide.
Three points are linearly dependent when, and only when, they are collinear.
Four points are linearly dependent when, and only when, they are complanar.
Five or more points are always linearly dependent.

Another geometrical application is suggested by the following considerations:

A set of n ordinary* quantities is nothing more or less than a complex quantity with n principal units. Our first definition of linear dependence is therefore precisely equivalent to the following:

The m complex quantities

$$a_1, a_2, \ldots . a_m$$

are said to be linearly dependent if m ordinary quantities $c_1, c_2, \ldots . c_m$ not all zero, exist such that:

$$c_1 a_1 + c_2 a_2 + \ldots . + c_m a_m = 0.$$

Now the simplest geometrical interpretation for a complex quantity with n principal units is as a vector in space of n dimensions,† and we are thus led to the conception of the linear dependence of vectors. The geometrical meaning of this linear dependence will be seen from the following theorems for the case $n = 3$.

Two vectors are linearly dependent when, and only when, they are collinear.
Three vectors are linearly dependent when, and only when, they are complanar.
Four or more vectors are always linearly dependent.

In order to get a geometrical interpretation of the linear dependence of functions, we must consider, not the functions themselves, but the equations obtained by equating these functions to zero. We speak of these equations as being linearly dependent if the functions are linearly dependent. If then we regard the independent variables as rectangular coordinates these equations give us geometrical loci in space of as many dimensions as there are independent variables. Thus in the cases of two and three variables, we have plane curves and surfaces respectively. The case of two loci is of no interest, as they must coincide in order to be linearly dependent. The following theorems will serve to illustrate the geometrical meaning of linear dependence.

(1.) In the plane:

Three circles are linearly dependent when, and only when, they belong to the same coaxial family.
Four circles are linearly dependent when, and only when, they have a (real or imaginary) common orthogonal circle.
Four circles are linearly dependent when, and only when, the points of intersection, of the first and second, and the points of intersection of the third and fourth, lie on a circle.

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* Two different standpoints are here possible according as we understand the term, *ordinary quantity* to mean, *real quantity*, or *ordinary complex quantity*.

† There are of course other possible geometrical interpretations. Thus in the case $n = 4$ we may regard our complex quantities as quaternions, and consider the meaning of linear dependence of two, three, or four quaternions.

Five or more circles are always linearly dependent.

(2.) In space:
Three planes are linearly dependent when, and only when, they intersect in a line.
Four planes are linearly dependent when, and only when, they intersect in a point.
Five or more planes are always linearly dependent.

2. The Condition for Linear Dependence of Sets of Constants. In considering the m sets of n constants each

$$x_1^{[i]}, x_2^{[i]}, \ldots x_n^{[i]} \qquad (i = 1, 2, \ldots m),$$

it will be convenient to distinguish between the two cases $m \leqq n$ and $m > n$.

(a) $m \leqq n$. We wish here to prove the following fundamental theorem:

The necessary and sufficient condition for linear dependence is that all the m-rowed determinants of the matrix

$$\begin{Vmatrix} x_1' & x_2' & \ldots & x_n' \\ x_1'' & x_2'' & \ldots & x_n'' \\ \ldots & \ldots & \ldots & \ldots \\ x_1^{[m]} & x_2^{[m]} & \ldots & x_n^{[m]} \end{Vmatrix}$$

should vanish.

That this is a necessary condition is at once obvious, for if the m sets of constants are linearly dependent, one of the rows can be expressed as a linear combination of the others. Accordingly if in any of the m-rowed determinants we subtract from the elements of this row the corresponding elements of the other rows, each row multiplied by a suitable constant, the elements of this row will reduce to zero. The determinant therefore vanishes.

We come now to the proof that the vanishing of these determinants is also a sufficient condition. We assume, therefore, that all the m-rowed determinants of the above matrix vanish. Let us also assume that at least one of the k-rowed determinants of the matrix does not vanish but that all determinants of order higher than k vanish.* Without any real loss of generality we may (and will) assume that *the k-rowed determinant which stands in the upper left hand corner of the matrix does not vanish;* for by changing the order of the sets of constants and the order of the constants in each set (and these orders are clearly quite immaterial) we can bring one of the non-vanishing k-rowed determinants into this position.

* In general we shall therefore have $k = m - 1$, but k may have any value less than m. The only case which we here exclude is that in which all the elements of the matrix are zero, a case in which the linear dependence is at once obvious.

We will now prove that the first $k+1$ sets of constants are linearly dependent. From this the linear dependence of the m sets follows by theorem II of the last section.

Let us denote by $c_1, c_2, \ldots . c_{k+1}$ the first minors of the $(k+1)$-rowed determinant which stands in the upper left hand corner of the matrix, and which correspond to the elements of its last column. If now we remember that all the $(k+1)$-rowed determinants vanish we get the relations:

$$c_1 x_j' + c_2 x_j'' + \ldots + c_{k+1} x_j^{[k+1]} = 0 \qquad (j = 1, 2, \ldots n).$$

This formula establishes the linear dependence of the first $k+1$ sets of constants. For $c_{k+1} \neq 0$, being the k-rowed determinant which stands in the upper left hand corner of the matrix.

(*b*) $m > n$. This case can be reduced to the one already considered by the following simple device. Add to each set of n constants $m-n$ zeros. We then have m sets of m constants each. Their matrix contains only one m-rowed determinant, which vanishes since one, at least, of its columns is composed of zeros. Therefore these m sets of m constants each are linearly dependent; and therefore the original m sets of n constants each were linearly dependent. Thus we get the theorem:

m sets of n constants each are always linearly dependent if $m > n$.

3. Linear Dependence of Polynomials. Linear Homogeneous Equations. Suppose we have m polynomials

$$f_1, f_2, \ldots . f_m$$

in any number of independent variables. The necessary and sufficient condition for the linear dependence of these polynomials is obviously the linear dependence of their m sets of coefficients.* Thus the conditions deduced in the last section can be applied at once to the case of polynomials.

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* It may be noticed that the number of independent variables involved in the polynomials is quite immaterial. What is important is the number of terms. Thus whether we have to deal with the three planes:

$$a_1 x + b_1 y + c_1 z + d_1 = 0,$$
$$a_2 x + b_2 y + c_2 z + d_2 = 0,$$
$$a_3 x + b_3 y + c_3 z + d_3 = 0;$$

or with the three circles:

$$a_1(x^2 + y^2) + b_1 x + c_1 y + d_1 = 0,$$
$$a_2(x^2 + y^2) + b_2 x + c_2 y + d_2 = 0,$$
$$a_3(x^2 + y^2) + b_3 x + c_3 y + d_3 = 0,$$

In the theory of simultaneous homogeneous linear equations we have a simple and extremely important application of the results so far obtained. We presuppose here a knowledge of the method of solving by determinants a set of n non-homogeneous linear equations in n unknowns when the determinant of the coefficients of the unknowns is not zero.*

We consider the system of equations:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &= 0,\\
a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n &= 0,\\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot &\\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot &\\
a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n &= 0.
\end{aligned}$$

The values $x_1 = x_2 = \cdots\cdots = x_n = 0$ satisfy these equations in all cases. We shall however in future discard this solution as being trivial, and shall understand by a solution of the equations a set of values of the x's satisfying the equations and not all zero.

The questions we have to answer are first whether the equations have a solution (in the sense just explained); and second how all the solutions which exist can be found. We will take up in succession the cases $m < n$, $m \geqq n$.

(a) $m < n$. Here we have the theorem:

The number of equations being less than the number of unknowns the equations always have solutions.

This is at once obvious if we consider the n sets of m constants each:

$$a_{1i},\ a_{2i},\ \cdots\cdots\ a_{mi} \qquad (i = 1, 2, \ldots\ n).$$

---

the condition for linear dependence is the vanishing of all the three rowed determinants of the matrix

$$\begin{Vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{Vmatrix}.$$

*This method will be found in every book where determinants are touched upon. The treatment, however, in most cases, is incomplete, all that is really proved being that *if the equations have a solution* it is given by the formulæ in question. To complete the treatment it is necessary to prove that the formulæ in question really satisfy the equations, as can readily be done by direct substitution.

These sets of constants are necessarily linearly dependent since $m < n$; *i. e.* there exist n constants $c_1, c_2, \ldots . c_n$, not all zero such that:

$$c_1 a_{j1} + c_2 a_{j2} + \ldots . + c_n a_{jn} = 0 \qquad (j = 1, 2, \ldots . m).$$

i. e. the c's satisfy the above system of linear equations.

(*b*) $m \geqq n$

The number of equations being equal to or greater than the number of unknowns, the necessary and sufficient condition for the existence of solutions is the vanishing of all the n-rowed determinants in the matrix of the coefficients of the equations.

The proof of this theorem is precisely like that of the preceding one.

We come now to the question of actually finding the solutions in the cases in which they exist, and we will begin with the case in which we have fewer equations than unknowns ($m < n$) and in which the equations are linearly independent. There will then be at least one m-rowed determinant in the matrix of coefficients which does not vanish, and we will assume for the sake of clearness that this is the determinant consisting of the first m-columns of the matrix. Let us now write the equations in the form:

$$\begin{aligned}
a_{11} x_1 + \ldots . + a_{1m} x_m &= - a_{1\,m+1} x_{m+1} - \ldots . - a_{1n} x_n, \\
a_{21} x_1 + \ldots . + a_{2m} x_m &= - a_{2\,m+1} x_{m+1} - \ldots . - a_{2n} x_n, \\
&\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
&\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
a_{m1} x_1 + \ldots . + a_{mm} x_m &= - a_{m\,m+1} x_{m+1} - \ldots . - a_{mn} x_n.
\end{aligned}$$

Assigning to $x_{m+1}, x_{m+2}, \ldots x_n$ any values whatever, we can compute from these equations the values of $x_1, x_2, \ldots x_m$; for we have then m non-homogenous equations in m unknowns in which the determinant of the coefficients of the unknowns does not vanish. We thus get the theorem:

If $m < n$ and the equations are linearly independent, to $n - m$ of the unknowns may be assigned arbitrary values and the others will then be completely determined.

It should be noticed that this theorem does not say that *any* $n - m$ of the unknowns can be chosen at pleasure, but merely that a certain set of $n - m$ unknowns can be so chosen,—in fact it is obvious that any set of $n - m$ unknowns may be so chosen which is such that the determinant formed from the coefficients of the remaining m unknowns is not zero.

In all the other cases in which the equations have solutions the equations are linearly dependent, and, even if there are more equations than there are unknowns, all the n-rowed determinants of the matrix of the coefficients vanish. We will now assume that there is at least one k-rowed determinant (which for the sake of clearness we assume as lying in the upper left hand corner of the matrix) which does not vanish, but that all determinants of the matrix of higher order vanish. Denoting then by $f_1, f_2, \ldots f_m$ the linear polynomials which stand on the left hand side of the equations, we see that the $k+1$ functions:

$$f_1, f_2, \ldots f_k, f_l,$$

(where l is any one of the integers $k+1, k+2, \ldots m$) are linearly dependent:

$$c_1 f_1 + c_2 f_2 + \cdots + c_k f_k + c_l f_l \equiv 0.$$

The constant c_l here is not zero as otherwise $f_1, f_2, \ldots f_k$ would be linearly dependent. We may therefore, by dividing through by c_l, express f_l linearly in terms of $f_1, f_2, \ldots f_k$. Accordingly any set of x's which satisfy the first k equations also satisfy the remaining equations. The first k equations however form a system of the sort already discussed, and we get the following theorem which we state in such a form as to include the last theorem stated as a special case:

If the system of equations admits a solution, and if in the matrix of the coefficients at least one k-rowed determinant does not vanish while all determinants of higher order (if there are any) are zero, then $n-k$ of the x's can be chosen at pleasure and the others are then completely determined.

Here again any $n-k$ x's can be chosen at pleasure which have the property that if their coefficients are struck out of the matrix at least one of the k-rowed determinants of the remaining matrix does not vanish.

The only case which is not covered by this theorem is that in which all the coefficients of all the equations are zero, in which case *all* the x's can of course be chosen at pleasure.

In the special case in which $m = n-1$ and the equations are linearly independent we readily get the following theorem:

The solutions of $n-1$ linearly independent homogeneous linear equations in n unknowns are proportional to the $(n-1)$-rowed determinants of the matrix of the coefficients obtained by striking out first the first column, then the second, etc., and taken alternately with the plus and minus sign.

We have here considered only homogeneous equations. Many facts concerning non-homogeneous linear equations can be deduced at once from the theorems here given by letting $x_n = 1$. For a complete treatment of the theory of non-homogeneous linear equations we refer to E. Pascal's admirable little book on determinants §56 (Hoepli, Milan; also a German translation by Leitzmann, Teubner, Leipzig).

4. The Condition for Linear Dependence of Functions of One Variable. Let us consider the n functions

$$f_1(x),\ f_2(x),\ \ldots\ .\ f_n(x).$$

We suppose these functions to be defined either throughout a certain region of the complex x-plane, or throughout a certain interval of the real x-axis. In the first case we assume that the functions are analytic throughout the region in question, and therefore have derivatives of all orders at every point of the region; in the second case we assume that, at every point of the interval, each of the functions has a finite derivative of each of the first $n - 1$ orders.

If the functions are linearly dependent we have not merely the relation, where the c's are constants not all zero:

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n \equiv 0,$$

but also by differentiation:

$$\begin{aligned}
c_1 f_1' &+ c_2 f_2' &&+ \cdots + c_n f_n' &&\equiv 0,\\
c_1 f_1'' &+ c_2 f_2'' &&+ \cdots + c_n f_n'' &&\equiv 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
c_1 f_1^{[n-1]} &+ c_2 f_2^{[n-1]} &&+ \cdots + c_n f_n^{[n-1]} &&\equiv 0.
\end{aligned}$$

These relations may be regarded, for each particular value of x, as a system of n homogeneous linear equations satisfied by the n c's. Accordingly the determinant of the coefficients of the c's must be zero; and since this is true for all the values of x with which we have to deal, we have the identity:

$$\begin{vmatrix}
f_1 & f_2 & \cdots & f_n\\
f_1' & f_2' & \cdots & f_n'\\
\cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots\\
f_1^{[n-1]} & f_2^{[n-1]} & \cdots & f_n^{[n-1]}
\end{vmatrix} \equiv 0.$$

This determinant is known as the *Wronskian* of the n functions $f_1, f_2, \ldots\ . f_n$.

We see therefore that the identical vanishing of the Wronskian is a necessary condition for linear dependence.

We will now consider under what circumstances it is also a sufficient condition. We begin with the case of two functions f_1, f_2 whose Wronskian vanishes identically:

$$f_1 f_2' - f_2 f_1' \equiv 0.$$

We will suppose at first that one of these functions, say f_1, does not vanish at any point of the region or interval in question. Then we can divide the above identity by f_1^2, after which it can be written:

$$\frac{d}{dx}\left(\frac{f_2}{f_1}\right) \equiv 0.$$

Therefore:

$$f_2 \equiv c f_1.$$

In the case of two functions, the identical vanishing of the Wronskian is a sufficient condition for linear dependence provided one of the two functions does not vanish at any point of the region in question.

This last restriction can be at once removed in the case that f_1 and f_2 are analytic functions of a complex variable. For if f_1 vanishes in the region in question, it either vanishes identically there, in which case the functions are necessarily linearly dependent, or it vanishes only at isolated points. In this latter case let us consider the region obtained from the original region, by omitting all the points at which f_1 vanishes. Throughout this new region we have, as has just been proved:

$$f_2 \equiv c f_1.$$

This relation holds throughout the neighborhood of each of the excluded points and therefore, on account of the continuity of f_1 and f_2, at these points, too; therefore:

The identical vanishing of the Wronskian of two analytic functions is a sufficient condition for linear dependence.

This is not true if we have to deal with non-analytic functions of a real variable, as the example

$$f_1 = x^2, \quad f_2 = x\,|x|$$

shows.* We will return to this case later, but proceed first to the general theorem concerning analytic functions:

The identical vanishing of the Wronskian of n analytic functions is a necessary and sufficient condition for their linear dependence.

We are concerned here merely with the proof that it is a sufficient condition. Let us assume that the theorem has been proved for the case of $n-1$ functions. If from this assumption we can deduce its proof in the case of n functions we shall have proved the theorem by the method of mathematical induction, since we know that it holds for two functions.

We start from the assumption that the Wronskian W of the n functions is identically zero. Let us denote the first minors of W which correspond to the elements of the last row by $W_1, W_2, \ldots W_n$. It should be noticed that W_i is the Wronskian of the $n-1$ functions obtained by omitting f_i. If then $W_i \equiv 0$ these $n-1$ functions are linearly dependent, and therefore (by theorem II, § 1) the n functions are linearly dependent.

Let us then assume that none of the minors W_i is identically zero. Now we have, by elementary theorems concerning determinants, the following identities:

$$\begin{aligned}
W_1 f_1 &+ W_2 f_2 + \cdots + W_n f_n \equiv 0,\\
W_1 f_1' &+ W_2 f_2' + \cdots + W_n f_n' \equiv 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
W_1 f_1^{[n-1]} &+ W_2 f_2^{[n-1]} + \cdots + W_n f_n^{[n-1]} \equiv 0.
\end{aligned}$$

Differentiating each of the first $n-1$ of these identities and subtracting the next following one we get the following new set of identities:

$$\begin{aligned}
W_1' f_1 &+ W_2' f_2 + \cdots + W_n' f_n \equiv 0,\\
W_1' f_1' &+ W_2' f_2' + \cdots + W_n' f_n' \equiv 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
W_1' f_1^{[n-2]} &+ W_2' f_2^{[n-2]} + \cdots + W_n' f_n^{[n-1]} \equiv 0.
\end{aligned}$$

* $|x|$ means the absolute or numerical value of x. This example is due to Peano, *Mathesis* vol. 9, p. 110 (1889). Even if we require the existence (and therefore continuity) of derivatives of all orders at every point of the interval the theorem does not hold. *Cf. Bull. Amer. Math. Soc.*, ser. 2, vol. 7, p. 120.

Let us multiply the ith of these identities ($i = 1, 2, \ldots n-1$) by the first minor of W_n corresponding to $f_1^{[i-1]}$ and add the identities thus obtained together. We get:

$$W_1' W_n - W_1 W_n' \equiv 0.$$

Now, the f's being analytic, the W's will be analytic, and therefore we can infer from this last written identity that W_1 and W_n are linearly dependent. In precisely the same way we see that any two of the functions W_i ($i = 1, 2, \ldots n$) are linearly dependent. Since by hypothesis none of these functions are identically zero we may write:

$$W_i \equiv c_i W_n \qquad (i = 1, 2, \ldots n-1).$$

The identity

$$W_1 f_1 + W_2 f_2 + \cdots + W_n f_n \equiv 0$$

can therefore be written:

$$W_n (c_1 f_1 + c_2 f_2 + \cdots + c_{n-1} f_{n-1} + f_n) \equiv 0.$$

Since W_n is not identically zero it can vanish only at isolated points. At all other points the other factor must vanish, and therefore, owing to the continuity of the f's, it must vanish at these points, too:

$$c_1 f_1 + c_2 f_2 + \cdots + c_{n-1} f_{n-1} + f_n \equiv 0.$$

The linear dependence of the f's is thus established and our theorem is proved.

Turning now to the case of non-analytic functions of a real variable we will first prove the following preliminary theorem for the case of two functions:

The identical vanishing of the Wronskian of two functions of a real variable, each of which has a finite first derivative at every point of a certain interval, is a sufficient condition for linear dependence, provided that one of the functions and its first derivative do not vanish together at any point of the interval in question.

The interval with which we have here to deal may be infinite in one or both directions, and if limited the limiting point or points may or may not belong to the interval. Our theorem however will obviously be established if we can prove it for every finite subinterval each of whose extremities is included in the subinterval. Let ab be such a subinterval, and suppose that f_1 and f_1' do not vanish together at any point of ab.

There cannot be more than a finite number of points in the interval ab where f_1 vanishes, for if there were they would have at least one limiting point x' in this interval. At the point x' f_1, being continuous (since it has a derivative), would vanish, and $f_1'(x') = \lim f(x' + \Delta x)/\Delta x$ would also vanish. This however is contrary to hypothesis.

The interval ab consists therefore of a finite number of pieces within each of which f_1 does not vanish. Within each of these pieces we therefore have a relation of the form:

$$f_2 \equiv c f_1,$$

and owing to the continuity of f_1 and f_2 this relation must also hold at the extremities of the piece in question. It remains therefore merely to prove that the constant c is the same for the different pieces; and this will be proved if we can show that it is the same for any two adjacent pieces.

Suppose then that x_0 is any point where $f_1 = 0$, and that in the piece to the right of x_0

$$f_2 \equiv c_1 f_1,$$

in the piece to the left

$$f_2 \equiv c_2 f_1.$$

Since f_1 and f_2 each has a derivative at x_0, these derivatives can be found either by differentiating to the right or to the left. We thus get:

$$f_2'(x_0) = c_1 f_1'(x_0),$$
$$f_2'(x_0) = c_2 f_1'(x_0),$$

and by subtraction:

$$(c_1 - c_2) f_1'(x_0) = 0.$$

But since f_1 vanishes at x_0, f_1' cannot vanish there, therefore: $c_1 = c_2$ and our theorem is proved.

The general theorem, of which the one just proved is a special case, is the following:

*The identical vanishing of the Wronskian of n functions of a real variable, each of which has at every point of a certain interval finite derivatives of the first $n - 1$ orders, is a sufficient condition for linear dependence, provided that $n - 1$ of the functions can be so selected that their Wronskian and its first derivative do not vanish together at any point of the interval.**

*A theorem different from this is proved by Peano: *Rend. d. Accad. d. Lincei*, ser. 5, vol. 6, 1° sem., p. 413.

As the proof of this theorem is almost identical with the proof given above in the case of analytic functions we will not give it here in detail. It will be convenient in adapting the above proof to the present case to assume that the Wronskian which does not vanish together with its first derivative is the one which is there denoted by W_n. Since this function cannot vanish identically the step in the above proof involving mathematical induction will not here come in. On the other hand the assumption which we made before that the other Wronskians W_i do not vanish identically, and which we do not now make, was not in any way made use of in the course of the proof.

5. Applications. We will indicate, in the present paragraph, a number of applications of the subject of linear dependence to various questions, leaving the reader to work out proofs for himself.

(1). In a system of m homogeneous linear equations, between n unknowns, one at least of the k-rowed determinants of the matrix of the coefficients does not vanish, while all $(k+1)$-rowed determinants (if any) are zero. Prove that the equations have $n-k$ linearly independent solutions, upon which every other solution is linearly dependent.

(2). DEFINITION: If we have a family of functions, of any number of variables, which can be expressed in the form:

$$k_1 f_1 + k_2 f_2 + \cdot \cdot \cdot \cdot + k_n f_n,$$

the constants k being allowed to take on all values, the family is called a *linear family* and the functions $f_1, f_2, \cdot \cdot \cdot f_n$, if they are linearly independent, are called a *basis* of the family.

Prove that if a linear family has a basis consisting of n functions:

(a) any set of $n+1$ functions of the family are linearly dependent.

(b) no set of less than n functions can form a basis.

(c) every set of n linearly independent functions of the family forms a basis.

Obtain the most general basis from the given one.

(3) DEFINITION: A spherical harmonic is a homogeneous polynomial in the three independent variables x, y, z which satisfies Laplace's equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Prove that the spherical harmonics of degree n form a linear family with a basis consisting of $2n+1$ functions.

(4). The fundamental existence theorem for linear differential equations, when the independent variable is supposed real, is:

If throughout a certain interval $p_1, p_2, \ldots . p_n$ are continuous functions of the real variable x, the equation

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + p_n y = 0$$

has a solution which has finite derivatives of the first n orders at every point of the interval in question, and which, at an arbitrarily chosen point of this interval has together with its derivatives of the first $n-1$ orders arbitrarily chosen values.

Assuming the truth of this theorem prove:

(a) The Wronskian of any n solutions of the equation has the value:

$$Ce^{-\int p_1 dx},$$

where C is a constant.

(b) The solutions of the equation form a linear family with a basis of n functions.

(c) The necessary and sufficient condition for the linear dependence of n solutions of the equation is the identical vanishing of the Wronskian; or, if one prefers, the vanishing of the Wronskian at some point of the interval.

(d) A solution of the equation which together with its first $n-1$ derivatives has at a point of the interval the value zero vanishes identically.

(5) Definition: the m sets of n integers each

$$x_1^{[i]}, x_2^{[i]}, \ldots . x_n^{[i]} \qquad (i = 1, 2, \ldots m)$$

are linearly dependent (mod p) if m integers $c_1, c_2, \ldots . c_m$ not all congruent to zero (mod p) exist such that:

$$c_1 x_j' + c_2 x_j'' + \cdots + c_m x_j^{[m]} \equiv 0. \qquad (\text{mod } p).$$

Assuming the modulus p to be prime, prove the following theorem:

The above sets of integers are always linearly dependent (mod p) if $m > n$. If $m \leqq n$ the necessary and sufficient condition for linear dependence (mod p) is that all m-rowed determinants in the matrix of the sets of integers be congruent to zero (mod p).

By means of this theorem discuss the theory of simultaneous homogeneous

linear congruences according to a prime modulus. (The general theory of simultaneous linear congruences has been discussed by Henry Smith in two papers *Phil. Trans.*, vol. 151, p. 293 (1861) and *Proc. London Math. Soc.*, vol. 4, p. 236 (1873); reprinted in the collected works, vol. 1, p. 367 and vol. 2, p. 67).

GÖTTINGEN, GERMANY, NOVEMBER, 1900.

BRILLIANT POINTS OF A FAMILY OF CONCENTRIC SPHERES.

By Alston Hamilton.

A brilliant point on the surface of a sphere is a point at which a ray of light, emanating from a fixed source, will be reflected to meet the eye of an observer. The problem is to construct the locus of such points for a family of concentric spheres.

Assume a point, O, in space as the centre of the spheres. Assume arbitrarily the points E and D, also in space, either of which may be the source of light and the other the point of vision. Through O, E, and D pass a plane. This plane will contain the brilliant point of every sphere for the given points D and E.

Assume OX and OY at right angles to each other. Use these as axes of X and Y respectively. Let the coordinates of E be (a, b) and those of D, (c, d). Assume a circle whose radius is r and centre, O. The equation of this circle is $x^2 + y^2 = r^2$. This circle is the one cut from one of the spheres by the XY plane. Select a point A on the circumference of this circle and draw AD and AE. Draw the normal and the tangent at A. Let the coordinates of A be (x', y'). Now, in order that A shall be the brilliant point for this sphere EAC must be equal to CAD; expressing this relation analytically we are led to the equation:

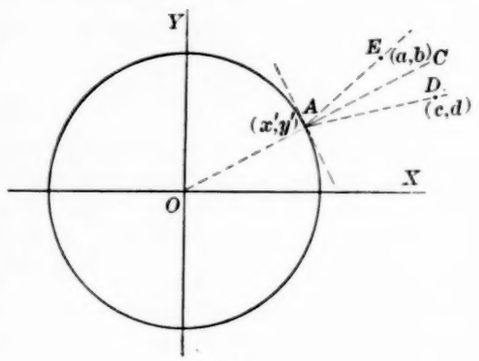

Fig. 1.

$$(b+d)(x^3+xy^2)-(a+c)(y^3+yx^2)+2(ac-bd)xy+(ad+bc)(y^2-x^2)=0 \qquad (1)$$

where x' and y' have been replaced by x and y.

This is the equation satisfied by the coordinates of the brilliant points and also by the coordinates of the so-called "virtual brilliant points." A is a *virtual brilliant point* if the tangent at A to the sphere bisects the angle EAD.

The equation is perfectly general, but unsatisfactory for tracing the curve. The following method has been employed to simplify the process while detracting nothing from its applicability to any case.

Let the coordinate axes be turned through an angle α such that in the new system the coefficient of $(y^2 - x^2)$ will disappear. α is given by the equation

$$\tan 2\alpha = \frac{ad + bc}{ac - bd} . \tag{2}$$

The angle 2α is thus seen to be equal to the sum of the angles that the right lines drawn through the origin to the points D and E make with the axis of X. Equation (1) becomes

$$x^3 + xy^2 - M(x^2y + y^3) + Qxy = 0. \tag{3}$$

Let (A, B) and (C, D) respectively be the coordinates of E and D referred to the new axes. Then

$$\left.\begin{aligned} A &= a\cos\alpha + b\sin\alpha, & C &= c\cos\alpha + d\sin\alpha, \\ B &= b\cos\alpha - a\sin\alpha, & D &= d\cos\alpha - c\sin\alpha, \\ M &= \frac{A + C}{B + D}, & Q &= \frac{2(AC - BD)}{B + D} . \end{aligned}\right\} \tag{4}$$

Now transform the equation (3) to polar coordinates having the new axis of X (or X') as initial line. The equation then becomes

$$r = \frac{Q\sin\theta}{M\tan\theta - 1} . \tag{5}$$

To trace the curve by means of (5) it is necessary to consider only values of θ between 0 and π, since an addition of π to θ changes the sign of r and hence gives no new point of the curve. From (5)

$$\frac{dr}{d\theta} = \frac{Q(M\sin\theta - \cos\theta - M\sin\theta\sec^2\theta)}{(M\tan\theta - 1)^2} . \tag{6}$$

Place $u = 1/r$; then

$$\frac{du}{d\theta} = \frac{M\sin\theta}{Q\cos^2\theta} + \frac{\cos\theta}{Q\sin^2\theta}$$

and

$$u + \frac{d^2u}{d\theta^2} = \frac{2M}{Q\cos^3\theta} - \frac{2}{Q\sin^3\theta} . \tag{7}$$

For a maximum radius vector

$$\frac{dr}{d\theta} = 0\,; \quad \text{whence, from (6),} \quad \tan\theta = \frac{-1}{\sqrt[3]{M}}\,; \quad r = \frac{-Q}{(1+M^{\frac{2}{3}})^{3/2}}\,. \tag{8}$$

For the asymptotes place

$$\infty = r = \frac{Q\sin\theta}{M\tan\theta - 1}\,; \quad \text{whence} \quad \tan\theta = \frac{1}{M}\,. \tag{9}$$

But

$$\frac{r^2 d\theta}{dr} = \frac{-Q\sin\theta}{M\tan^2\theta + \cot\theta} = \frac{-MQ}{(M^2+1)^{3/2}},$$

which is finite and represents the distance of the origin from the asymptote. Hence there is always an asymptote. To draw it, draw a line through the origin making an angle with the initial line $\tan^{-1} 1/M$, and draw another line through the origin at right angles to this. On this second line, lay off the distance $r^2 d\theta/dr$ to the right (looking from the origin in the direction of the line first drawn) if $r^2 d\theta/dr$ is positive, or to the left if $r^2 d\theta/dr$ is negative. Through the extremity of this line, draw a line parallel to the line first drawn. It will be an asymptote.

For a point of inflexion,

$$u + \frac{d^2 u}{d\theta^2} = 0,$$

and from (7),

$$\tan\theta = \frac{1}{\sqrt[3]{M}}, \qquad r = \frac{Q}{(M^{\frac{1}{3}}-1)\sqrt{M^{\frac{2}{3}}+1}}\,.$$

When ϕ is the angle made by the tangent at the point of inflection with the radius vector

$$\tan\phi = \frac{r d\theta}{dr} = -\frac{1}{2}(M^{\frac{1}{3}} - M^{-\frac{1}{3}}).$$

For a multiple point,

$$f = 0, \quad \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0,$$

where $f(x, y)$ represents the first member of equation (3). These equations are satisfied only at the origin, and the two tangents at this point are evidently $x = 0$, $y = 0$ corresponding to the values $\theta = 0$ and $\frac{\pi}{2}$.

Example. Let D be $(-6, 2)$ and E be $(4, 7)$ referred to the first system of coordinates. Then $2a = \tan^{-1}\frac{7}{10}$, $A = 5.1$, $B = -6.25$, $C = 4$,

$D = 4.9$, $M = -6.75$, $Q = -75.6$. From (5), $r = \dfrac{75.6 \sin\theta}{6.75\tan\theta + 1}$. For a maximum radius vector $\tan\theta = .53$, $r = 7.7$. For the asymptote, $\tan\theta = -.15$, $r^2\, d\theta/dr = 1.61$. For the point of inflexion $\tan\theta = -.526$, $r = 13.8$, $\tan\phi = .69$

The above points having been determined, the following geometric method will save labor in determining other points. From each of the points D and E draw two tangents to the circle BAC arbitrarily assumed with centre at O, but not including D or E. These tangents will be ER, EB, DT, DA and will intersect in four points F, G, H, K. The four points thus found will be real or virtual brilliant points, as is at once evident from a glance at the figure (Fig. 2). By varying the circle BAC any number of such points may be found. Draw EO and DO and prolong them. The angles EOW and DOS include all *virtual* brilliant points and the angles EOD and WOS include all *real* brilliant points.

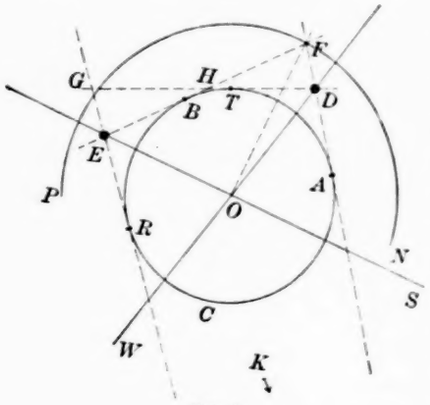

FIG. 2.

Figure 3 shows the description by curve-tracing methods of the example given. It is to be noticed that the asymptote is parallel to the tangents drawn from the eye and light respectively to the circle for which the tangents will be parallel to each other. The direction of the asymptote may thus be checked graphically. The above is readily seen from the description of the curve by points by the graphical method.

Figure 4 shows the description of the same curve by the graphical method. In both figures the part of the curve containing virtual brilliant points is dotted. A combination of the two methods, using the graphical method for intermediate points gives more satisfactory results than either alone.*

* Extract from Lieutenant Hamilton's letter to the Editor: "The problem as discussed would seem to indicate that use may be made of its law, analytically expressed, in constructing *range-finders*. Since solving it I have, indeed, planned from it several varieties of range-finder; but with a small instrument the base is too small to give very accurate results, and difficulties in the way of sufficiently accurate construction abound to such an extent as to disappoint me with the result."

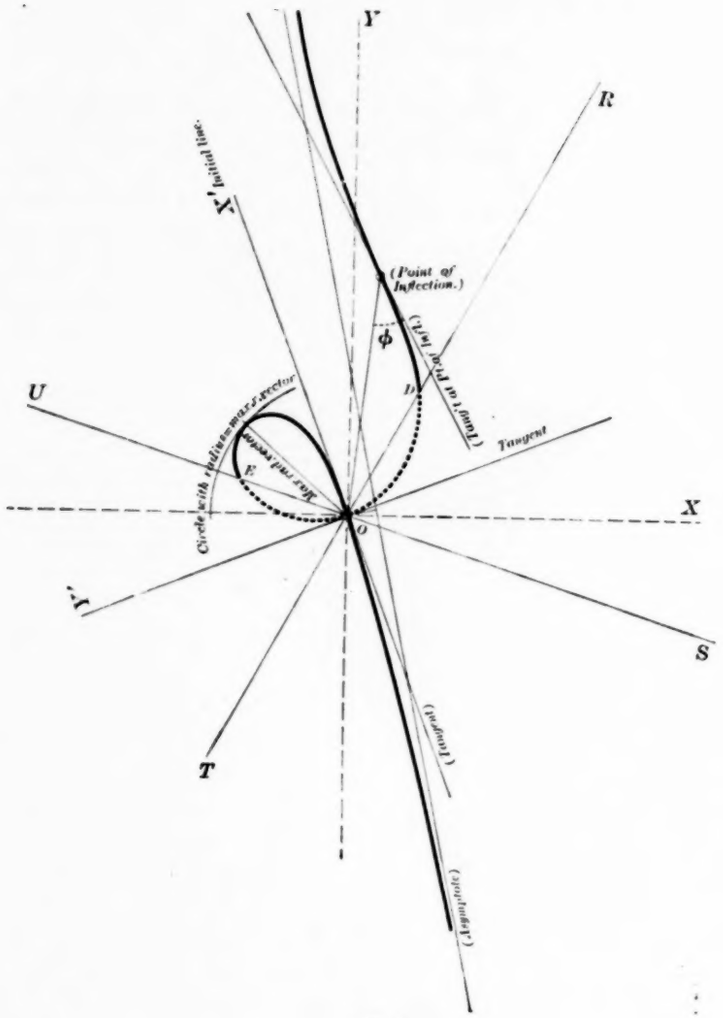
Y
R
X' Initial line.
(Point of Inflection.)
ϕ
(Path.)
Tangent at Point (Path.)
U
Tangent
Circle with radius=unit vector
Unit vector
E
X
O
Y'
S
T
(Tangent)
(Asymptote)

FIG. 3.

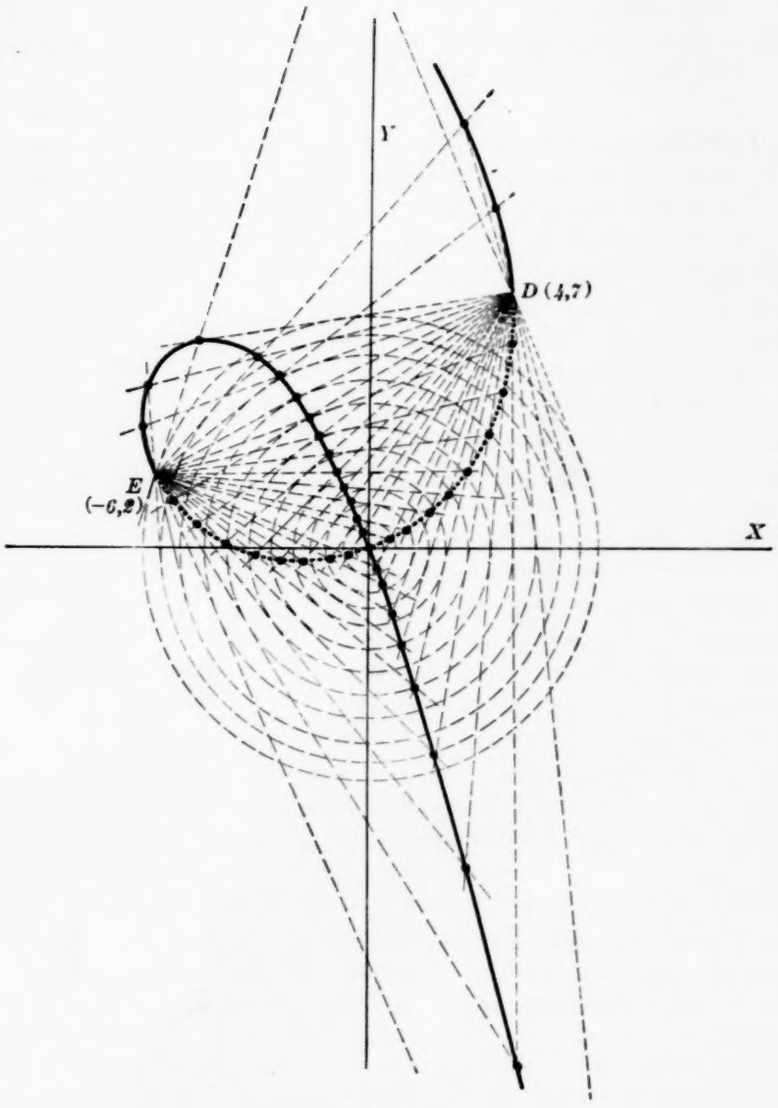

FIG. 4.

ANGELES, LUZON, PHILIPPINE ISLANDS, APRIL 12, 1900.

MULTIPLY PERFECT NUMBERS.

By Derrick N. Lehmer.

1. By a *perfect number* is meant a number which is equal to the sum of those of its divisors which are less than the number. Thus $6 = 1 + 2 + 3$ is a perfect number.

We shall define a *multiply perfect number* as one which is an exact divisor of the sum of all its divisors; the quotient being the *multiplicity*. Thus ordinary perfect numbers like 6 or 28 have the multiplicity 2.

The existence of multiply perfect numbers of higher multiplicity than 2 is shown by the following list.

NUMBER.	MULTIPLICITY.
$2^3 \cdot 3 \cdot 5$	3
$2^5 \cdot 3 \cdot 7$	3
$2^9 \cdot 3 \cdot 11 \cdot 31$	3
$2^{13} \cdot 3 \cdot 11 \cdot 43 \cdot 127$	3
$2^{14} \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151$	3
$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	4
$2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$	4
$2^5 \cdot 3^3 \cdot 5 \cdot 7$	4
$2^7 \cdot 3^3 \cdot 5^2 \cdot 17 \cdot 31$	4
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$	4
$2^8 \cdot 3^2 \cdot 7^2 \cdot 13 \cdot 19^2 \cdot 37 \cdot 73 \cdot 127,$	4
$2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31,$	4
$2^{14} \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151,$	4
$2^7 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 17 \cdot 19,$	5
$2^{10} \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19 \cdot 23 \cdot 89,$	5
$2^{11} \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31,$	5
$2^{21} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 19 \cdot 23^2 \cdot 31 \cdot 79 \cdot 89 \cdot 137 \cdot 547 \cdot 683 \cdot 1093,$	5
$2^{20} \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 13^2 \cdot 19 \cdot 31 \cdot 61 \cdot 127 \cdot 337,$	5
$2^{19} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19 \cdot 23 \cdot 31 \cdot 41 \cdot 137 \cdot 547 \cdot 1093,$	6
$2^{24} \cdot 3^8 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19^2 \cdot 31 \cdot 43 \cdot 53 \cdot 127 \cdot 379 \cdot 601 \cdot 757 \cdot 1801,$	6

The numbers in the above list were all obtained by purely tentative

methods. No formula for multiply perfect numbers has been obtained except the well-known one for those of multiplicity 2 (ordinary perfect numbers), namely, $2^{i-1}(2^i-1)$, where 2^i-1 is prime.*

2. If $\kappa = \prod_{i=1}^{r} p_i^{a_i}$ is a perfect number of multiplicity n, then we have

$$n\kappa = \prod_{i=1}^{r} \sum_{j=0}^{a_i} p_i^j = \prod_{i=1}^{r} \frac{p_i^{a_i+1}-1}{p_i-1}$$

or

$$n = \prod_{i=1}^{r} \frac{p_i - \frac{1}{p_i^{a_i}}}{p_i-1};$$

so that

$$n < \prod_{i=1}^{r} \frac{p_i}{p_i-1}. \qquad (1)$$

Sylvester† has derived this formula for the case $n = 2$. It is of fundamental importance in his proof of the non-existence of odd perfect numbers containing only three distinct prime divisors.

3. The object of the present note is a generalization of Sylvester's results for the case of multiply perfect numbers.

Multiply perfect numbers of multiplicity 3, *containing less than three distinct primes, do not exist.* For, from (1) it follows that $3 < \prod_{i=1}^{r} \frac{p_i}{p_i-1}$, and the greatest value which this product can have when there are only two elements is $\frac{2}{1} \cdot \frac{3}{2}$, which is not greater than 3.

Similarly: *Multiply perfect numbers of multiplicity* 4 *must contain at least* 4 *distinct prime divisors*; *those of multiplicity* 5, *at least* 6 *distinct prime divisors*; *those of multiplicity* 6, *at least* 9 *distinct prime divisors; those of multiplicity* 7, *at least* 14 *distinct prime divisors; etc.*

BERKELEY, CAL., JANUARY, 1900.

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* Given by Euclid and shown by Euler to include all even perfect numbers. *Cf. Encyclopädie der mathematischen Wissenschaften*, vol. 1, art. C 1, Bachmann.

† *Comptes Rendus*, vol. 106 (1888), pp. 403, 446, 522, 641.

GALOIS' THEORY OF ALGEBRAIC EQUATIONS.

SIX LECTURES DELIVERED AT THE BUFFALO COLLOQUIUM IN SEPTEMBER, 1896.

BY

PROFESSOR JAMES PIERPONT.

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